THE BEHAVIOR OF A SELF-OSCILLATING SYSTEM ACTED UPON BY SLIGHT NOISE

(O RABOTE AVTOKOLEBATEL'NOI SISTEMY PRI VOZDEISTVII NA NEE MALOGO SHUMA)

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We consider the behavior of a self-oscillating system

$$X'' + \omega^2 X' - \varepsilon f(X, X') = \mu \xi'(t) \tag{0.1}$$

for small values of ε and μ, where $\xi'(t)$ is a "white noise" process. We shall investigate the probability density of a transition (Section 2) and of a stationary distribution (Section 3) of a Markov process $(X(t)$ $X'(t))$, defined by equation (0.1), under various assumptions regarding the order of magnitude of $\mu/\sqrt{\varepsilon}$. In particular, it is shown that if $\mu/\sqrt{\varepsilon} \ll 1$, then the "white noise" may be neglected in calculating the steady state of self-oscillations. Particular attention is paid to the case $\mu/\sqrt{\varepsilon} \sim 1$. It is shown that in this case the stationary probability distribution tends to a limit as $\varepsilon \to 0$. This limit is found. The effective frequency of the oscillations is calculated (Section 4) to within a quantity which is $o(\varepsilon)$. The results are applied to the Van der Pol case (Section 5). In this particular case the stationary distribution is found to be Gaussian.

1. As is known, the system $F(x'', x', x, \varepsilon) = 0$ (which is conservative for $\varepsilon = 0$) has a stable limit cycle for any arbitrarily small ε, if F satisfies certain conditions. The methods for calculating the position of this limit cycle for small values of ε have been developed in detail. These methods, dating back to Van der Pol, were established on a more general basis in the works of N.N. Bogoliubov and N.N. Krylov (the averaging principle).

It may happen, however, that as $\varepsilon \to 0$, the system will become sensitive to small random disturbances which "spread out" its limiting operation. It is demonstrated below that the operation of such a system may be analyzed with the aid of a theorem, proved by the author, which extends the averaging principle to systems with random noise.

Equation (0.1) may be written more correctly in the form of stochastic differential equations [1, p.248]

$$dX(t) = Y(t)\,dt,$$
$$dY(t) = [-\omega^2 X(t) + \varepsilon f(X(t), Y(t))]\,dt + \sqrt{\varepsilon}\,\sigma d\xi(t) \qquad \left(\sigma = \frac{\mu}{\sqrt{\varepsilon}}\right) \quad (1.1)$$

Here $\xi(t)$ is a Wiener random process (that is, a process with independent increments and a Gaussian probability distribution; in this case* $\langle \xi(t) \rangle = 0$, $\langle \xi^2(t) \rangle = t$). The solution of the system (1.1), as is known [1], is a time-uniform Markov process $(X(t), Y(t))$ in the phase space of the system. We shall denote by $p_\varepsilon(x, y, t, x_0, y_0)$ the probability density of a transition from the point (x, y) to the point (x_0, y_0) in time t for the trajectory of this process. This density, as a function of x, y, t, satisfies equation

$$\frac{\partial p}{\partial t} = y\frac{\partial p}{\partial x} - \omega^2 x \frac{\partial p}{\partial y} + \varepsilon\left[\sigma^2 \frac{\partial^2 p}{\partial y^2} + f(x, y)\frac{\partial p}{\partial y}\right] \quad (1.2)$$

and the initial condition $p_\varepsilon(x, y, 0, x_0, y_0) = \delta(x - x_0, y - y_0)$.

The density for a stationary distribution of this process, $P_\varepsilon(x_0, y_0)$, defined by equations

$$P_\varepsilon(x_0, y_0) = \int P_\varepsilon(x, y)\; p_\varepsilon(x, y, t, x_0, y_0)\,dxdy \quad (1.3)$$

$$\int P_\varepsilon(x_0, y_0)\,dx_0 dy_0 = 1 \quad (1.4)$$

satisfies equation

$$\varepsilon\left[\sigma^2\frac{\partial^2 P_\varepsilon}{\partial y^2} - \frac{\partial}{\partial y}(f(x, y)P_\varepsilon)\right] - y\frac{\partial P_\varepsilon}{\partial x} + \omega^2 x\frac{\partial P_\varepsilon}{\partial y} = 0 \quad (1.5)$$

Naturally, a stationary operation of system (1.1), and hence also a function P_ε, satisfying conditions (1.3) and (1.4), will not necessarily exist for every function $f(x, y)$. In the remainder of this article it will be assumed that the function f satisfies conditions such that $P_\varepsilon(x, y)$, which is a solution of the problem (1.5) and (1.4), exists and that for a fixed value of σ the function $P_\varepsilon(x, y)$ does not "spread out" as $\varepsilon \to 0$; this is equivalent to the condition: for any $\delta > 0$, there exists some $R > 0$ such that for all $\varepsilon > 0$ we have

* Here and hereafter, pointed brackets will be used to denote probability averaging.

$$\int\limits_{r<R} P_\varepsilon(x, y)\, dx\, dy \geqslant 1 - \delta \qquad (r = \sqrt{x^2 + y^2}) \tag{1.6}$$

2. Changing to new coordinates in equation (1.2), we readily obtain the equation for the function

$$q_\varepsilon(r, \varphi, t, r_1, \varphi_1) \equiv p_\varepsilon([r/\omega]\sin\varphi, r\cos\varphi, t, [r_1/\omega]\sin\varphi_1, r_1\cos\varphi_1)$$

Introducing another unknown function

$$u_\varepsilon(r, \varphi, t, r_1, \varphi_1) = q_\varepsilon(r, \varphi - \omega t, t, r_1, \varphi_1) \tag{2.1}$$

which is equivalent to changing to a rotating coordinate system $x = (r/\omega)\sin(\varphi - \omega t)$, $y = r\cos(\varphi - \omega t)$, we obtain the equation

$$\frac{\partial u_\varepsilon}{\partial t} = \varepsilon\Big\{\sigma^2\Big[\cos^2(\varphi - \omega t)\frac{\partial^2 u_\varepsilon}{\partial r^2} - \frac{\sin 2(\varphi - \omega t)}{r}\frac{\partial^2 u_\varepsilon}{\partial r \partial\varphi} + \frac{\sin^2(\varphi - \omega t)}{r^2}\frac{\partial^2 u_\varepsilon}{\partial\varphi^2} + \frac{\sin 2(\varphi - \omega t)}{r^2}\frac{\partial u_\varepsilon}{\partial\varphi} + \frac{\sin^2(\varphi - \omega t)}{r}\frac{\partial u_\varepsilon}{\partial r}\Big] + f\Big(\frac{r}{\omega}\sin(\varphi - \omega t), r\cos(\varphi - \omega t)\Big)\Big[\cos(\varphi - \omega t)\frac{\partial u_\varepsilon}{\partial r} + \frac{\sin(\varphi - \omega t)}{r}\frac{\partial u_\varepsilon}{\partial\varphi}\Big]\Big\} \tag{2.2}$$

Let x be a point of n-dimensional Euclidean space. The averaging principle has been proved [2,3] for differential equations of the form $\partial u/\partial t = \varepsilon L(x, t)u$, where L is an elliptical or parabolic second-order differential operator; according to this principle the solution of the Cauchy problem for this equation as $\varepsilon \to 0$ may be uniformly approximated over an interval of time which is $O(1/\varepsilon)$ by the solution of the equation $\partial v/\partial t = \varepsilon L^\circ(x)v$, where $L^\circ(x)$ is an operator whose coefficients are obtained from those of $L(x, t)$ by averaging with respect to time, that is

$$L^\circ(x) = \lim_{T\to\infty}\frac{1}{T}\int_0^T L(x, t)\, dt$$

Applying this averaging principle to equation (2.2) and taking account of (2.1), we obtain the following result: let $p_0(r, \varphi, t, r_1, \varphi_1)$ be the probability density of a transition of the random process to a plane described in polar coordinates by equation

$$\frac{\partial p_0}{\partial t} = \frac{\sigma^2}{2}\Big[\frac{\partial^2 p_0}{\partial r^2} + \frac{1}{r}\frac{\partial p_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p_0}{\partial\varphi^2}\Big] + \Phi(r)\frac{\partial p_0}{\partial r} + \frac{\Psi(r)}{r}\frac{\partial p_0}{\partial\varphi} \tag{2.3}$$

where

$$\Phi(r) = \frac{1}{2\pi}\int_0^{2\pi} f\left(\frac{r}{\omega}\cos t,\ r\sin t\right)\sin t\,dt,\ \Psi(r) = \frac{1}{2\pi}\int_0^{2\pi} f\left(\frac{r}{\omega}\cos t,\ r\sin t\right)\cos t\,dt \quad (2.4)$$

Then for any $R > 0$ and $T > 0$

$$q_\varepsilon(r, \varphi - \omega t, t, r_1, \varphi_1) - p_0(r, \varphi, t\varepsilon, r_1, \varphi_1) \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad (2.5)$$

uniformly with respect to r, φ, r_1, φ_1 in the region $r < R$, $r_1 < R$ and with respect to t in the region $0 \leqslant t \leqslant T/\varepsilon$.

The relation (2.5) may also be rewritten as

$$q_\varepsilon(r, \varphi, t, r_1, \varphi_1) - p_0(r, \varphi + \omega t, t\varepsilon, r_1, \varphi_1) \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad (2.6)$$

which is more suitable for our further work.

3. If, for the process described by equation (2.3), there exists a stationary density $p(r, \varphi)$, then it will evidently be independent of φ and will be a solution of the problem

$$\frac{\sigma^2}{2}[(pr)'' - p'] - (\Phi(r)\,p)' = 0, \qquad \int_0^\infty p(r)\,r\,dr = 1$$

Hence, we find

$$p(r) = \left[\int_0^\infty \exp\left\{\frac{2}{\sigma^2}\int_0^r \Phi(s)\,ds\right\} r\,dr\right]^{-1} \exp\left\{\frac{2}{\sigma^2}\int_0^r \Phi(s)\,ds\right\} \quad (3.1)$$

It is assumed that the function $\Phi(r)$ satisfies the conditions under which the integral in (3.1) will converge; the convergence of this integral, as is known, is necessary and sufficient for stationary operation in the process described by equation (2.3). We introduce the notation

$$Q_\varepsilon(r, \varphi) = P_\varepsilon(r\omega^{-1}\sin\varphi,\ r\cos\varphi)$$

We shall prove that for any bounded function $f(r, \varphi)$

$$\iint f(r, \varphi)\,Q_\varepsilon(r, \varphi)\,r\,dr\,d\varphi \to \iint f(r, \varphi)\,p(r)\,r\,dr\,d\varphi \quad \text{as} \quad \varepsilon \to 0 \quad (3.2)$$

From known limit theorems on Markov processes with invariant measure [4, 5] it follows that for any $\delta > 0$ and $R > 0$ there exists a T_0 such that

$$\left|\iint f(r_1, \varphi_1)\,[p_0(r, \varphi, T_0, r_1, \varphi_1) - p(r_1)]\,r_1\,dr_1\,d\varphi_1\right| < \delta \quad \text{where } r < R \quad (3.3)$$

From (2.6), (1.6) and (3.3) we obtain the inequality

$$\left|\iint f(r_1, \varphi_1)\left[q_\varepsilon\left(r, \varphi, \frac{T_0}{\varepsilon}, r_1, \varphi_1\right) - p(r_1)\right] r_1\, dr_1\, d\varphi_1\right| < 2\delta \quad \text{if} \quad \varepsilon \leqslant \varepsilon_0(\delta, T_0), \quad r < R \tag{3.4}$$

From (3.4), taking account of the identity

$$\iint f(r, \varphi)\, Q_\varepsilon(r, \varphi)\, r\, dr\, d\varphi = \iint Q_\varepsilon(r, \varphi)\, r\, dr\, d\varphi \iint q_\varepsilon\left(r, \varphi, \frac{T_0}{\varepsilon}, r_1, \varphi_1\right) f(r_1, \varphi_1)\, r_1\, dr_1\, d\varphi_1$$

we find that for $\varepsilon \leqslant \varepsilon_0$

$$\left|\iint f Q_\varepsilon\, r\, dr\, d\varphi - \iint f p r\, dr\, d\varphi\right| \leqslant$$

$$\leqslant \Big(\iint_{r<R} + \iint_{r \geqslant R}\Big) Q_\varepsilon(r, \varphi)\, r\, dr\, d\varphi \left|\iint\left[q_\varepsilon\left(r, \varphi, \frac{T_0}{\varepsilon}, r_1, \varphi_1\right) - p(r_1)\right] f(r_1, \varphi_1)\, r_1 dr_1 \varphi_1\right| \leqslant$$

$$\leqslant 2\delta + M \iint_{r \geqslant R} Q_\varepsilon(r, \varphi)\, r\, dr\, d\varphi \qquad (M = \sup|f|)$$

Since $\delta > 0$ and $R > 0$ are arbitrary, it follows that, taking (1.6) into account, we obtain (3.2).

Equation (3.2) enables us to investigate the behavior of the stationary measure of the process (1.1) as $\varepsilon \to 0$ for different orders of magnitude of $\sigma = \mu/\sqrt{\varepsilon}$, since it is sufficient for this to investigate the behavior of the function $p(r)$ defined by equation (3.1). Evidently for the density of the distribution $rp(r)$ with respect to the measure $dr d\varphi$ we have extremum points where

$$\Phi(r) = -\sigma^2 / 2r \tag{3.5}$$

In the limiting case $\sigma \to 0$ equation (3.5) becomes the well-known equation for the equilibrium points of an oscillatory system which approximates a harmonic oscillator [6, p. 658]

$$\Phi(r) = 0 \tag{3.6}$$

This result shows that in the case of noise power $\mu^2 \ll \varepsilon$, white noise may be neglected in the study of the oscillations.

In the second limiting case, $\sigma \to \infty$, the stationary distribution "spreads out". This means that for $\mu^2 \gg \varepsilon$, no stable oscillatory behavior is possible.

Physically, the quantity

$$E_\varepsilon(t) = \tfrac{1}{2}\{\omega^2 [X_\varepsilon(t)]^2 + [Y_\varepsilon(t)]^2\}$$

represents the energy of the oscillations at time t. From ergodic theorems it follows that the average energy of the oscillations over time T as $T \to \infty$ has for almost all trajectories the limit

$$\langle E_\varepsilon \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_0^T E_\varepsilon(t)\, dt = \frac{1}{2} \int r^2 P_\varepsilon(r, \varphi)\, r dr d\varphi$$

We assume that the following condition (somewhat more restrictive than (1.6)) is satisfied: for all $\varepsilon > 0$ and fixed σ

$$\iint\limits_{r>R} P_\varepsilon(r, \varphi)\, r^3 dr d\varphi < \delta \quad \text{for } R \geqslant R_0(\delta)$$

Then from (3.2) we find

$$\lim_{\varepsilon\to 0} \langle E_\varepsilon \rangle = \langle E_0 \rangle = \frac{1}{2} \int_0^\infty r^3 p(r)\, dr \tag{3.7}$$

4. Let $\zeta(t)$ be a random variable equal to the number of times the component $X_\varepsilon(t)$ of the process $(X_\varepsilon(t), Y_\varepsilon(t))$ goes through zero from left to right. As is known, the limit of the random variable $(1/t)\zeta(t)$ as $t \to \infty$ is, with a probability of unity, $\lim [(1/t) < \zeta(t) >]$, where the averaging is taken for any initial distribution. This limit multiplied by 2π is called the effective frequency of oscillation of the process, $< \omega_\varepsilon >$.

It is clear that for $\varepsilon = 0$, $\sigma = 0$, the quantity $< \omega_\varepsilon >$ is identical with ω. The presence of nonlinearity and random disturbances necessitates a corrective term added to the frequency. It is known that in the absence of noise ($\sigma = 0$, $\varepsilon \neq 0$) the formula

$$\omega_\varepsilon = \omega + \varepsilon \frac{\Psi(r_0)}{r_0} + o(\varepsilon) \qquad (\varepsilon \to 0) \tag{4.1}$$

is valid, where $\Psi(r)$ is defined by equation (2.4) and the constant r_0 is determined from equation (3.6). The purpose of the present section is to obtain the analog of the formula (4.1) for any σ.

In order to calculate $< \omega_\varepsilon >$ it is convenient to consider the random process $(X_\varepsilon(t), Y_\varepsilon(t))$ in another phase space so that the number of times that the trajectory of the process has encircled zero will be "remembered".

The mapping inverse to $x = r\omega^{-1} \sin \varphi$, $y = r \cos \varphi$ translates the Markov process $X_1 = (X_\varepsilon(t), Y_\varepsilon(t))$ in the xy-plane into the Markov process $X_2 = (r(t), \varphi(t))$ on the set $K = \{0 \leqslant \varphi < 2\pi,\ r > 0\}$, where (r, φ) and $(r, \varphi + 2\pi)$ represent the same point.

Each trajectory $(r(t), \varphi(t))$ of the process X_2 will be mapped onto the trajectory of a new process $X_3 = (\rho(s), \theta(s))$ in the half-plane $K_1(-\infty < \theta < \infty, \rho > 0)$ by the formulas

$$\rho(s) = r(s), \qquad \theta(s) = \varphi(s) + 2\pi\zeta(s)$$

and we shall require this mapping to preserve the probability measure on the set of trajectories. It is easy to verify that the process X_3 thus constructed is also a Markov process. Its transition probability density $q_\varepsilon^*(\rho, \theta, t_1, \rho_1, \theta_1)$ (with respect to the measure $\rho_1 d\rho_1 d\theta_1$) satisfies the same differential equation in the variables ρ, θ, t as the function $q_\varepsilon(r, \varphi, t, r_1, \varphi_1)$ does in the variables r, φ, t. However, unlike the function q_ε, which is the Green's function of this equation on the set $(K, t > 0)$, the function q_ε^* is the Green's function on $(K_1, t > 0)$.

Applying the method of Section 2 (it must be proved that the theorems from [2, 3] are applicable to this situation), we can again obtain equation (2.6), with q_ε replaced by q_ε^* and p_0 replaced by p_0^*, where p_0^* is the Green's function of equation (2.3) on the set $(K_1, t > 0)$ (not $(K, t > 0)$, as is the case with p_0). Furthermore, just as in Section 3 we found equation (3.2) from (2.6), we can find from this the relation

$$\langle\omega_\varepsilon\rangle = \omega + \varepsilon V(1) + o(\varepsilon) \qquad (\varepsilon \to 0) \tag{4.2}$$

for

$$\langle\omega_\varepsilon\rangle = \lim_{T\to\infty} \frac{1}{T} \int\limits_{\rho=0}^{\infty} \int\limits_{\theta=-\infty}^{\infty} q_\varepsilon^*(\rho, \theta, T, \rho_1, \theta_1)(\theta_1 - \theta)\rho_1 d\rho_1 d\theta_1$$

Here

$$V(t) = \int\limits_{r=0}^{\infty} p(r)\, r dr \int\limits_{r_1=0}^{\infty} \int\limits_{\theta_1=-\infty}^{\infty} p_0^*(r, \theta, t, r_1, \theta_1)(\theta_1 - \theta)\, r_1 dr_1 d\theta_1$$

Multiplying (2.3) by $\theta_1 d\theta_1 r_1 dr_1$ and integrating, we find that the function

$$u_0(r, \theta, t) = \iint\limits_{K_1} p_0^*(r, \theta, t, r_1, \theta_1)\theta_1 r_1 dr_1 d\theta_1$$

also satisfies equation (2.3) on $(K_1, t > 0)$ and the initial condition $u_0(r, \theta, 0) = \theta$.

It follows from this that $v_0(r, \theta, t) = u_0(r, \theta, t) - \theta$ will be the solution of the problem

$$\frac{\partial v_0}{\partial t} = \frac{\sigma^2}{2}\left[\frac{\partial^2 v_0}{\partial r^2} + \frac{1}{r}\frac{\partial v_0}{\partial r}\right] + \Phi(r)\frac{\partial v_0}{\partial r} + \frac{\Psi(r)}{r}, \qquad v_0(r, 0) = 0$$

Multiplying the last equation by $rp(r)dr$ and integrating, we find

$$\frac{dV}{dt} = \int \Psi(r)\, p(r)\, dr, \qquad V(0) = 0$$

and consequently

$$V(t) = t \int \Psi(r)\, p(r)\, dr$$

Substituting this value of $V(t)$ into (4.2), we finally obtain

$$\langle \omega_\varepsilon \rangle = \omega + \varepsilon \int_0^\infty \Psi(r)\, p(r)\, dr + o(\varepsilon) \qquad (\varepsilon \to 0) \tag{4.3}$$

5. Let us consider an example for which

$$\omega = 1, \qquad f(x, y) = y(1 - x^2) \tag{5.1}$$

In the absence of noise (for $\mu = 0$) we obtain the Van der Pol equation, for which, as is known

$$\Phi(r) = {}^1/_2 r - {}^1/_8 r^3, \qquad \Psi(r) = 0 \tag{5.2}$$

Applying the conclusions of Sections 2 to 4, we obtain from (5.2), (3.1) and (4.3)

$$p(r) = \left[2\sqrt{\pi}\,\sigma \exp\left(\frac{1}{\sigma^2}\right) F\left(\frac{\sqrt{2}}{\sigma}\right)\right]^{-1} \exp\left[\frac{1}{\sigma^2}\left(\frac{r^2}{2} - \frac{r^4}{16}\right)\right]$$

$$\left(F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{y^2}{2}\right) dy\right)$$

$$\langle \omega_\varepsilon \rangle = \omega + o(\varepsilon) \qquad (\varepsilon \to 0)$$

It should be noted that the applicability of the conclusions of Sections 2 to 4 in the present case requires additional proof, since in [2,3] it was assumed that the coefficients of the equation increase to infinity no faster than linear functions, while $f(x, y) = y(1 - x^2)$ does not satisfy this condition.

From (3.5) it is clear that in the present case the function $rp(r)$ has a single maximum at the point $r_0 = [2 + 2\surd(1 + \sigma^2)]^{1/2}$

Hence, as $\sigma \to 0$, we obtain the well-known approximate value for the radius of the limit cycle in the no-noise case: $r_0 = 2$. The average energy of the oscillations in this example as $\varepsilon \to 0$ tends to a limit (see (3.7)) is

$$\langle E_0 \rangle = 2 + (1 / \sqrt{\pi})\, \sigma \exp(-1/\sigma^2)\, [F(\sqrt{2}/\sigma)]^{-1}$$

In conclusion, it should be noted that the effect of random noise on the operation of a self-oscillating system of the type considered here was studied in [7-9]. However, in all of those investigations it was assumed that the noise power was much less than the parameter characterizing the nonlinearity (that is, $\sigma \ll 1$ in the notation of the present study). It is readily seen that the results obtained for $\sigma \ll 1$ agree with the results of [7-8].

We should mention that the method used here is suitable for investigating the effect of random noise on more general systems, both one-dimensional and multi-dimensional, which contain a small parameter ε and become conservative for $\varepsilon = 0$.

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